Available online at www.sciencedirect.com
ScienceDirect
Discrete Applied Mathematics 156 (2008) 2835–2845
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

The profile of the Cartesian product of graphs

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Received 8 October 2006; received in revised form 15 November 2007; accepted 21 November 2007
Available online 4 January 2008

Abstract

Given a graph G, a *proper labeling* f of G is a one-to-one function from $V(G)$ onto $\{1, 2, \ldots, |V(G)|\}$. For a proper labeling f of G, the *profile width* $w_f(v)$ of a vertex v is the minimum value of $f(v) - f(x)$, where x belongs to the closed neighborhood of v. The *profile of a proper labeling* f *of* G, denoted by $P_f(G)$, is the sum of all the $w_f(v)$, where $v \in V(G)$. The *profile of* G is the minimum value of $P_f(G)$, where f runs over all proper labeling of G. In this paper, we show that if the vertices of a graph G can be ordered to satisfy a special neighborhood property, then so can the graph $G \times Q_n$. This can be used to determine the profile of Q_n and $K_m \times Q_n$.

Keywords: Profile; n-cube; Optimal labeling; Cartesian product; PMNL-graph; P-property

1. Introduction

The profile minimization problem was introduced by Gibbs and Poole [2] as a technique for handling sparse matrices. Given a sparse symmetric $n \times n$ matrix A, suppose for each row i, $a_{ii} \neq 0$ and t_i is the position for the first nonzero element in this row. We call $w_i = i - t_i = i - \min\{j \mid a_{ij} \neq 0\}$ the *width* of row i, and call

$$P(A) = \sum_{i=1}^{n} w_i$$

the *profile* of matrix A. To store A, we only have to store $w_i + 1$ elements in each row i, which are from position t_i to position i. The total amount of storage for this scheme is then $P(A) + n$. In order to reduce the amount of storage, we only have to permute the rows and columns of A simultaneously so that the resulting matrix has minimum profile, i.e., we need to find a permutation matrix Q so that $P(Q^{-1}AQ)$ is minimized.

Reformulating this problem in terms of graphs, we consider the following labeling problem of graphs. Given a graph G, a *proper labeling* of G is a one-to-one function f from $V(G)$ onto $\{1, 2, \ldots, |V(G)|\}$. For a proper labeling f, the *profile width* $w_f(v)$ of a vertex v in a graph G is

$$w_f(v) = \max_{x \in N[v]} (f(v) - f(x)).$$

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0166-218X/$ - see front matter
doi:10.1016/j.dam.2007.11.016

The *profile* $P_f(G)$ *of a proper labeling* f *of* G is defined by

$$P_f(G) = \sum_{v \in V(G)} w_f(v),$$

and the *profile* of G is $P(G) = \min\{P_f(G) | f \text{ is a proper labeling of } G\}$. A proper labeling f of G is called an *optimal labeling* if $P_f(G) = P(G)$.

Lin and Yuan [11] showed that the profile minimization problem of an arbitrary graph is equivalent to the interval graph completion problem, which was shown to be an NP-complete problem by Garey and Johnson [3]. Kuo and Chang [6] provided a polynomial-time algorithm to achieve a profile numbering for an arbitrary tree of order n. Lai [8] gave exact profile values of some graph compositions. Lai and Chang [9] found the profiles of corona of two graphs $G \wedge H$, when G is a caterpillar, a complete graph or a cycle. Lai [7] gave an algorithm to find an optimal labeling of hypercubes. There are also many other results surrounding this topic. For a good survey, see [1,10].

Lin and Yuan [11] showed that for any proper labeling f of a graph $G = (V, E)$,

$$P_f(G) = \sum_{i=1}^{n} | N(f^{-1}(\{1, 2, \ldots, i\}))|,$$

where $N(S) = \{v \in V - S \mid uv \in E, \text{for some } u \in S\}$ for all $S \subseteq V$. We define $N[S] = N(S) \cup S$ and $n[k] = \min\{|N[S]| \mid S \subseteq V, |S| = k\}$, $k = 0, 1, 2, \ldots, |V|$. From these definitions, if f is a proper labeling of G which satisfies

$$|N[f^{-1}(\{1, 2, \ldots, k\})]| = n[k]$$

for all $1 \le k \le |V|$, then f is an optimal labeling of G. In this case, we call the labeling f a *minimum-neighborhood labeling*. A *perfect minimum-neighborhood labeling* is a minimum-neighborhood labeling in which

$$N[f^{-1}(\{1, 2, \ldots, k\})] = f^{-1}(\{1, 2, \ldots, n[k]\}).$$

A graph G is called a *PMNL-graph* if it has a perfect minimum-neighborhood labeling.

The *Cartesian product* of two graphs G and H, denoted by $G \times H$, is defined by

$$V(G \times H) = \{(u, v) \mid u \in V(G) \text{ and } v \in V(H)\}$$

and

$$E(G \times H) = \{(u, x)(v, y) \mid (u = v \text{ and } xy \in E(H)) \text{ or } (x = y \text{ and } uv \in E(G))\}.$$

Lin and Yuan [11] found the profile of the Cartesian product of a path with a path, and a path with a cycle. The profiles of the Cartesian products of a path or a cycle with a complete graph, and a cycle with a cycle were found by Mai [12].

An *n-cube* Q_n is a graph $G = (V, E)$ in which

$$V = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \{0, 1\}, i = 1, 2 \ldots, n\},$$
$$E = \left\{(x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n) \mid \sum_{i=1}^{n} |x_i - y_i| = 1\right\}.$$

It is easy to see that $Q_n = Q_{n-1} \times K_2$. There are two problems, the bandwidth and the bandwidth sum problems, which are closely related to the profile problem. Both of the bandwidth and bandwidth sum for Q_n were found by Harper [4,5]. In paper [5], Harper gave a method to construct a perfect minimum-neighborhood labeling (which he called Hales numbering) of Q_n. Thus, Q_n is a PMNL-graph. In Section 2, we prove that $G \times Q_n$ is a PMNL-graph for some special classes of graphs and construct a perfect minimum-neighborhood labeling for those graphs. In Section 3, we show that a labeling of $K_m \times Q_n$ that satisfies some special properties is a perfect minimum-neighborhood labeling and use it to determine the profile of $K_m \times Q_n$.

Throughout this paper, we let $V(G \times K_2) = V_1 \cup V_2$, where $V_i = \{(x, i) \mid x \in V(G)\}$, $i = 1, 2$, and let $G_i =< V_i >$, $i = 1, 2$. Note that, $G_1 \cong G_2 \cong G$.

2. Main theorem

Lemma 1. *Suppose S is a vertex subset of $G \times K_2$ with $|S| = k$ and $|N_{G\times K_2}[S]| = n_{G\times K_2}[k]$. If $|S \cap V_1| = j$ or $|S \cap V_2| = j$, then*

$$n_{G\times K_2}[k] \geq \max\{n_G[j] + n_G[k-j], n_G[j] + j, n_G[k-j] + k - j\}.$$

Proof. Assume $|S \cap V_1| = j$. Since $N_{G_i}[S \cap V_i] \subseteq N_{G\times K_2}[S] \cap V_i$ for $i = 1, 2$, we have

$$\begin{aligned} n_{G\times K_2}[k] &= |N_{G\times K_2}[S]| \\ &= |N_{G\times K_2}[S] \cap V_1| + |N_{G\times K_2}[S] \cap V_2| \\ &\geq |N_{G_1}[S \cap V_1]| + |N_{G_2}[S \cap V_2]| \\ &\geq n_G[j] + n_G[k-j]. \end{aligned}$$

On the other hand,

$$\begin{aligned} |N_{G\times K_2}[S]| &\geq |N_{G\times K_2}[S \cap V_1]| \\ &= |(N_{G\times K_2}[S \cap V_1] \cap V_1) \cup (N_{G\times K_2}[S \cap V_1] \cap V_2)| \\ &= |N_{G_1}[S \cap V_1]| + |S \cap V_1| \\ &\geq n_G[j] + j. \end{aligned}$$

Similarly, $|N_{G\times K_2}[S]| \geq |N_{G\times K_2}[S \cap V_2]| \geq n_G[k-j] + k - j$. Hence $n_{G\times K_2}[k] \geq \max\{n_G[j] + n_G[k-j], n_G[j] + j, n_G[k-j] + k - j\}$. The case of $|S \cap V_2| = j$ is similar. $\square$

Since $|V(G \times K_2)| = 2|V(G)|$. We want to give a sufficient condition so that for each integer $1 \leq k \leq 2|V(G)|$,

$$n_{G\times K_2}[k] = n_G[j] + j$$

for some j.

Let $G = (V, E)$ be a graph of order p. Partition the set $\{1, 2, \ldots, 2p\}$ into $A[0], A[1], \ldots, A[p]$, where $A[k] = \{a > 0 \mid n[k] + k \leq a \leq n[k+1] + k\}$ for all $k = 0, 1 \ldots, p-1$, and $A[p] = \{2p\}$. We say that G satisfies the *P-property* if for all $0 \leq k \leq p$, we have

$$n[t] + n[a-t] \geq n[a-k] + a - k$$

for each $a \in A[k]$ and $k < t < a - k$. Note that $a \geq 2k$ when $a \in A[k]$.

Lemma 2. *Suppose G satisfies the P-property. Then $n_{G\times K_2}[a] \geq n_G[a-k] + a - k$ for each $a \in A[k]$.*

Proof. Let S be a vertex subset of $G \times K_2$ with $|S| = a$ and $|N_{G\times K_2}[S]| = n_{G\times K_2}[a]$. Assume $|S \cap V_1| = t \leq \lfloor a/2 \rfloor$. Case 1. $t \leq k$: we have $|S \cap V_2| = a - t \geq a - k$. By Lemma 1, $n_{G\times K_2}[a] \geq n_G[a-t] + a - t \geq n_G[a-k] + a - k$. Case 2. $t > k$: we have $k < t \leq \lfloor a/2 \rfloor$, hence $t \leq a - t < a - k$. By Lemma 1 and the P-property, $n_{G\times K_2}[a] \geq n_G[t] + n_G[a-t] \geq n_G[a-k] + a - k$. $\square$

Lemma 3. *If $G = (V, E)$ is a PMNL-graph of order p and $a \in A[k]$, then*

$$n_{G\times K_2}[a] \leq n_G[a-k] + a - k.$$

Proof. Since $a \in A[k]$, we have $n[k] \leq a - k$. Let $V = \{v_1, v_2, \ldots, v_p\}$ and let $f(v_i) = i$ be a perfect minimum-neighborhood labeling of G. Then $N_G[\{v_1, v_2, \ldots, v_k\}] \subseteq \{v_1, v_2, \ldots, v_{a-k}\}$. Let $S = S_1 \cup S_2$, where $S_1 = \{(v_1, 1), (v_2, 1), \ldots, (v_{a-k}, 1)\}$ and $S_2 = \{(v_1, 2), (v_2, 2), \ldots, (v_k, 2)\}$. Then

$$\begin{aligned} n_{G\times K_2}[a] &\leq |N_{G\times K_2}[S]| \\ &= |N_{G_1}[S_1]| + |S_1| \\ &= n_G[a-k] + a - k. \quad \square \end{aligned}$$

By Lemmas 2 and 3, we have the following lemma.

Lemma 4. *Suppose G is a PMNL-graph that satisfies the P-property and $a \in A[k]$. Then*

$$n_{G\times K_2}[a] = n_G[a-k] + a - k,$$

and the set $S = S_1 \cup S_2$, where $S_1 = \{(v_1, 1), (v_2, 1), \ldots, (v_{a-k}, 1)\}$ and $S_2 = \{(v_1, 2), (v_2, 2), \ldots, (v_k, 2)\}$, satisfies $n_{G\times K_2}[a] = |N_{G\times K_2}[S]|$.

Let $G = (V, E)$ be a PMNL-graph of order p with $V = \{v_1, v_2, \ldots, v_p\}$ and let $f(v_i) = i$ be a perfect minimum-neighborhood labeling. The labeling g of $G \times K_2$ obtained from the following procedure DL is called a *direct labeling* with respect to f.

Procedure DL
Step 1. Let $i = j = k = 1$.
Step 2. If $j \leq p$ and $v_j \in N_G[v_i]$, then let $g(v_j, 1) = k$ and add one to k and j, and repeat step 2. Otherwise, let $g(v_i, 2) = k$ and add one to k.
Step 3. If $i < p$, then add one to i and go to step 2.

Theorem 5. *Suppose G is a PMNL-graph that satisfies the P-property. Then a direct labeling g of $G \times K_2$ is a perfect minimum-neighborhood labeling. Thus, $G \times K_2$ is a PMNL-graph.*

Proof. From procedure DL, it is easy to see that if $a \in A[k]$, then $S = g^{-1}(\{1, 2, \ldots, a\}) = S_1 \cup S_2$, where $S_1 = \{(v_1, 1), (v_2, 1), \ldots, (v_{a-k}, 1)\}$ and $S_2 = \{(v_1, 2), (v_2, 2), \ldots, (v_k, 2)\}$. So, $n_{G\times K_2}[a] = n_G[a-k] + a - k = |N_{G\times K_2}[S]|$. Since no labels greater than $n_G[a-k] + a - k$ are used when $i \leq a - k$ in Step 2 of Procedure DL, we have $N_{G\times K_2}[S] = g^{-1}(\{1, 2, \ldots, n_{G\times K_2}[a]\})$. Thus, g is a perfect minimum-neighborhood labeling of $G \times K_2$. □

Theorem 6. *Suppose G is a PMNL-graph that satisfies the P-property. Then $G \times K_2$ satisfies the P-property.*

Proof. We need only show that for each $j < l < m - j$,

$$n_{G\times K_2}[l] + n_{G\times K_2}[m-l] \geq n_{G\times K_2}[m-j] + m - j$$

when $m \in A_{G\times K_2}[j]$.

If $j < l \leq \lfloor \frac{m}{2} \rfloor$, then $\lceil \frac{m}{2} \rceil \leq m - l < m - j$. Without loss of generality, we may assume $j < l \leq \lfloor \frac{m}{2} \rfloor$. Let $l \in A_G[i_1]$, $m - l \in A_G[i_2]$, and $j \in A_G[t]$. It is easy to see that $t \leq i_1 \leq i_2$. By Lemma 4, we have

$$n_{G\times K_2}[l] = n_G[l - i_1] + l - i_1,$$
$$n_{G\times K_2}[m-l] = n_G[m - l - i_2] + m - l - i_2.$$

Thus,

$$n_{G\times K_2}[l] + n_{G\times K_2}[m-l] = n_G[l - i_1] + n_G[m - l - i_2] + m - (i_1 + i_2).$$

If we can show that $n_G[l - i_1] + n_G[m - l - i_2] \geq n_{G\times K_2}[m-j]$ and $i_1 + i_2 \leq j$, then we complete the proof.

Claim 1. $m - l - i_2 \geq l - i_1$.

Proof. If $i_1 = i_2$, then $m - l - i_2 = m - l - i_1 \geq l - i_1$. Assume $i_2 > i_1$. Since $l \in A_G[i_1]$ and $m - l \in A_G[i_2]$, we have

$$l - i_1 \leq n_G[i_1 + 1] \leq n_G[i_2] \leq m - l - i_2. \quad □$$

Claim 2. $j + 1 \in A_G[t]$ *or* $j + 1 = n_G[t+1] + t + 1$.

Proof. Since $j \in A_G[t]$, we have

$$n_G[t] + t \leq j \leq n_G[t+1] + t.$$

Thus, $j + 1 \in A_G[t]$ or $j + 1 = n_G[t+1] + t + 1$. □

Claim 3. *$n_G[j-t]+j-t \le m-j \le n_G[j-t+1]+j-t+1$. That is, $m-j \in A_G[j-t]$ or $m-j = n_G[j-t+1]+j-t+1$.*

Proof. Since $m \in A_{G\times K_2}[j]$, we have

$$n_{G\times K_2}[j] \le m-j \le n_{G\times K_2}[j+1].$$

By Lemma 4, $n_{G\times K_2}[j] = n_G[j-t]+j-t$. By Claim 2, $j+1 \in A_G[t]\cup A_G[t+1]$. If $j+1 \in A_G[t]$, then $n_{G\times K_2}[j+1] = n_G[j-t+1]+j-t+1$. So,

$$n_G[j-t]+j-t \le m-j \le n_G[j-t+1]+j-t+1.$$

Thus, $m-j \in A_G[j-t]$ or $m-j = n_G[j-t+1]+j-t+1$.

If $j+1 \in A_G[t+1]$, then $n_{G\times K_2}[j+1] = n_G[j-t]+j-t = n_{G\times K_2}[j]$. So $m-j = n_G[j-t]+j-t$. Thus, $m-j \in A_G[j-t]$. □

Claim 4. *$i_2 \le j-t$.*

Proof. By Claim 3, we have

$$n_G[j-t]+j-t \le m-j \le n_G[j-t+1]+j-t+1.$$

Since $l > j$ and $m-l \in A_G[i_2]$, $n_G[i_2]+i_2 \le m-l \le m-j-1 \le n_G[j-t+1]+j-t$. Thus, $i_2 \le j-t$. □

Claim 5. *$i_1+i_2 \le j$.*

Proof. Assume $i_1 = t$. By Claim 4, $i_1+i_2 \le t+(j-t) = j$.

For the case of $i_1 > t$, by Claim 3,

$$n_G[i_1]+n_G[i_2]+i_1+i_2 \le m = m-j+j \le n_G[j-t+1]+2j-t+1.$$

Assume $i_1+i_2 > j$. Then $n_G[i_1]+n_G[i_2] < n_G[j-t+1]+j-t+1$. Suppose $j+1 \in A_G[t]$, by Claim 4, $t < i_1 \le i_2 < j-t+1$. Since G satisfies the P-property, we have

$$\begin{aligned} n_G[i_1]+n_G[i_2] &\ge n_G[i_1]+n_G[j+1-i_1] \\ &\ge n_G[j-t+1]+j-t+1, \end{aligned}$$

a contradiction. Thus, $i_1+i_2 \le j$.

If $j+1 = n_G[t+1]+t+1$, then $m-j = n_G[j-t]+j-t$ by the proof of Claim 3. So,

$$n_G[i_1]+n_G[i_2]+i_1+i_2 \le m = m-j+j = n_G[j-t]+2j-t,$$

and $n_G[i_1]+n_G[i_2] < n_G[j-t]+j-t$. Since $l > j$, we have $n_G[i_2]+i_2 \le m-l \le m-j-1 = n_G[j-t]+j-t-1$. Thus, $i_2 \le j-t-1$ and $i_1 > t+1$. Since $j+1 \in A_G[t+1]$ and $t+1 < i_1 \le i_2 \le j-t-1 < j+1-(t+1)$, by the definition of the P-property,

$$\begin{aligned} n_G[i_1]+n_G[i_2] &\ge n_G[i_1]+n_G[j+1-i_1] \\ &\ge n_G[j+1-(t+1)]+j+1-(t+1) \\ &= n_G[j-t]+j-t, \end{aligned}$$

a contradiction. Thus, $i_1+i_2 \le j$. □

Claim 6. *$n_G[l-i_1]+n_G[m-l-i_2] \ge n_{G\times K_2}[m-j]$.*

Proof. Let $m-(i_1+i_2) \in A_G[d]$ and $k = j-(i_1+i_2)$. By Claim 3, either $m-j = n_G[j-t+1]+j-t+1$ or $m-j \in A_G[j-t]$. We consider the following cases.

Case 1. $m-j = n_G[j-t+1]+j-t+1$. Since

$$\begin{aligned} n_G[d]+d &\le m-(i_1+i_2) \\ &= m-j+k \\ &= n_G[j-t+1]+j-t+1+k \\ &\le n_G[j-t+1+k]+j-t+1+k, \end{aligned}$$

and the function $f(c) = n_G[c] + c$ is an increasing function, we have $d \le j - t + 1 + k = j - t + 1 + j - i_1 - i_2$. Thus,

$$i_1 + i_2 + d \le 2j - t + 1.$$

If $l - i_1 \le d$, then $l + i_2 \le i_1 + i_2 + d \le 2j - t + 1$. Since $i_2 \le j - t$ and $m - l \in A_G[i_2]$, we have

$$\begin{aligned} n_G[i_2 + 1] &\le n_G[j - t + 1] \\ &= m - (2j - t + 1) \\ &\le n_G[i_2 + 1] + i_2 + l - (2j - t + 1) \\ &\le n_G[i_2 + 1]. \end{aligned}$$

This implies all $n_G[i_2+1] = n_G[i_2+1]+i_2+l-(2j-t+1) = n_G[j-t+1]$. Thus, $l+i_2 = 2j-t+1 = i_1+i_2+d$ and $l - i_1 = d = 2j - (i_1 + i_2) - t + 1 \ge j - t + 1$. By Lemma 4, we have

$$\begin{aligned} n_G[l - i_1] + n_G[m - l - i_2] &\ge n_G[j - t + 1] + n_G[m - (2j - t + 1)] \\ &= n_G[m - 2j + t - 1] + m - 2j + t - 1 \\ &= n_{G\times K_2}[m - j]. \end{aligned}$$

For the case of $l - i_1 > d$, we have $l + i_2 > i_1 + i_2 + d$. Since $m - i_1 - i_2 \in A_G[d]$ and $d < l - i_1 \le m - l - i_2 < m - i_1 - i_2 - d$, by the P-property and Lemma 4, we have

$$\begin{aligned} n_G[l - i_1] + n_G[m - l - i_2] &= n_G[l - i_1] + n_G[m - i_1 - i_2 - (l - i_1)] \\ &\ge n_G[m - i_1 - i_2 - d] + m - i_1 - i_2 - d \\ &\ge n_G[m - 2j + t - 1] + m - 2j + t - 1 \\ &= n_{G\times K_2}[m - j]. \end{aligned}$$

Case 2. $m - j \in A_G[j - t]$ and $i_1 = t$. Suppose $l - t < m - 2j + t$. By Claim 4, the P-property, and Lemma 4, we have

$$\begin{aligned} n_G[l - i_1] + n_G[m - l - i_2] &\ge n_G[l - t] + n_G[m - l - (j - t)] \\ &= n_G[l - t] + n_G[m - j - (l - t)] \\ &\ge n_G[m - j - (j - t)] + m - j - (j - t) \\ &= n_{G\times K_2}[m - j]. \end{aligned}$$

If $l - t \ge m - 2j + t$, then $m - 2j + t \le l - t = l - i_1 \le m - l - i_2$ by Claim 1. Thus,

$$\begin{aligned} n_G[l - i_1] + n_G[m - l - i_2] &\ge n_G[m - 2j + t] + n_G[m - 2j + t] \\ &\ge n_G[m - j - (j - t)] + m - j - (j - t) \\ &= n_{G\times K_2}[m - j]. \end{aligned}$$

Case 3. $m - j \in A_G[j - t]$ and $i_1 > t$. Since

$$\begin{aligned} n_G[d] + d &\le m - (i_1 + i_2) \\ &= m - j + k \\ &\le n_G[j - t + 1] + j - t + k \\ &\le n_G[j - t + k + 1] + j - t + k, \end{aligned}$$

$d \le j - t + k = j - t + j - i_1 - i_2$. Thus,

$$i_1 + i_2 + d \le 2j - t.$$

Since $i_1 > t$, by Claim 5, $i_2 < j - t$.

If $l - i_1 \le d$, then $l + i_2 \le i_1 + i_2 + d \le 2j - t$. Since $i_2 \le j - t - 1$ and $m - l \in A_G[i_2]$, we have

$$\begin{aligned} n_G[i_2+1] &\le n_G[j-t] \\ &\le m - (2j - t) \\ &\le n_G[i_2+1] + i_2 + l - (2j-t) \\ &\le n_G[i_2+1]. \end{aligned}$$

This implies $n_G[i_2+1] = n_G[i_2+1] + i_2 + l - (2j-t) = n_G[j-t]$. Thus, $l + i_2 = 2j - t = i_1 + i_2 + d$ and $l - i_1 = d = 2j - (i_1 + i_2) - t \ge j - t$. If $l - i_1 > j - t$, then $n_G[l - i_1] \ge n_G[j - t + 1] \ge m - 2j + t$. Suppose $l - i_1 = j - t$. By Claim 2, $j - t \le n_G[t+1]$, therefore, $j - t \le n_G[t+1] \le n_G[i_1] \le l - i_1 = j - t$ and then $j + 1 = n_G[t+1] + t + 1$. From the proof of Claim 3, we have $m - j = n_G[j-t] + j - t$. Thus, $n_G[l - i_1] = n_G[j-t] = m - 2j + t$. By Lemma 4,

$$\begin{aligned} n_G[l-i_1] + n_G[m-l-i_2] &\ge n_G[j-t] + n_G[m-(2j-t)] \\ &\ge n_G[m-2j+t] + m - 2j + t \\ &= n_{G\times K_2}[m-j]. \end{aligned}$$

For the case of $l - i_1 > d$, we have $l + i_2 > i_1 + i_2 + d$. Since $m - i_1 - i_2 \in A_G[d]$ and $d < l - i_1 \le m - (l + i_2) < m - i_1 - i_2 - d$. By the P-property and Lemma 4, we have

$$\begin{aligned} n_G[l-i_1] + n_G[m-l-i_2] &= n_G[l-i_1] + n_G[m-i_1-i_2-(l-i_1)] \\ &\ge n_G[m-i_1-i_2-d] + m - i_1 - i_2 - d \\ &\ge n_G[m-2j+t] + m - 2j + t \\ &= n_{G\times K_2}[m-j]. \quad \square \end{aligned}$$

It is easy to see that K_m is a PMNL-graph with the P-property. Since $Q_1 \cong K_2$ and $Q_n \cong Q_{n-1} \times K_2$, we have that the graphs $K_m \times Q_n$ are PMNL-graphs with the P-property for all $m, n \ge 1$. In the next section, we give the profile of $K_m \times Q_n$.

3. Exact profile of $K_m \times Q_n$

In the section, we will find a perfect minimum-neighborhood labeling of $K_m \times Q_n$ and compute the profile of $K_m \times Q_n$. To simplify the notation, we use $Q_{m,n}$ to denote the graph $K_m \times Q_n$. We let

$$V(Q_{m,n}) = \{(x_1, x_2, \ldots, x_n, a) | x_i = 0, 1 \text{ and } a = 1, 2, \ldots, m\}$$

and $(x_1, x_2, \ldots, x_n, a)(y_1, y_2, \ldots, y_n, b) \in E(Q_{m,n})$ if and only if

$$(x_i = y_i \quad \text{for all } i = 1, 2, \ldots, n \text{ and } a \ne b) \quad \text{or} \quad \left(\sum_{i=1}^{n} |x_i - y_i| = 1 \text{ and } a = b\right).$$

Suppose $v = (z_1, \ldots, z_k, a) \in V(Q_{m,k})$. Let $x_1 \ldots x_r v y_1 \ldots y_s$ be the vertex $(x_1, \ldots, x_r, z_1, \ldots, z_k, y_1, \ldots y_s, a)$ of $Q_{m,r+k+s}$, and let $x_1 \ldots x_r Q_{m,k} y_1 \ldots y_s$ be the subgraph of $Q_{m,r+k+s}$ induced by the vertex subset

$$\{(x_1, \ldots, x_r, z_1, \ldots, z_k, y_1, \ldots y_s, a) \mid (z_1, \ldots, z_k, a) \in V(Q_{m,k})\}.$$

If $u = (u_1, u_2, \ldots, u_n, a)$, we let $\text{rank}(u) = \sum_{i=1}^{n} u_i$, and let $\text{right}(u) = a$. We define $R_{m,n}(k) = \{u \in V(Q_{m,n}) \,|\text{rank}(u) = k\}$, $T_{m,n}(a) = \{u \in V(Q_{m,n}) \,|\text{right}(u) = a\}$.

Since $Q_{1,n} \cong Q_{2,n-1}$, $n \ge 1$, we may assume $m \ge 2$ for $Q_{m,n}$ except $m = n = 1$ and vertex $(u_1, u_2, \ldots, u_n, 1)$ in $Q_{1,n}$ is the vertex $(u_1, u_2, \ldots, u_{n-1}, u_n + 1)$ in $Q_{2,n-1}$. The *exact labeling* of $Q_{m,n}$ is a proper labeling obtained from the following rules:

(1) The exact labeling f of $Q_{1,1}$ is defined by $f(0, 1) = 1$ and $f(1, 1) = 2$.
(2) The exact labeling f of $Q_{m,n}$ is a labeling so that $f(u) > f(v)$ if and only if $u = (u_1, u_2, \ldots, u_n, a)$ and $v = (v_1, v_2, \ldots, v_n, b)$ satisfy one of the following conditions:

(a) $\operatorname{rank}(u) > \operatorname{rank}(v)$.
(b) $\operatorname{rank}(u) = \operatorname{rank}(v)$ and $a > b$.
(c) $a = b$, and $g(u_1, u_2, \ldots, u_n, 1) > g(v_1, v_2, \ldots, v_n, 1)$, where g is the exact labeling of $Q_{1,n}$ (that is, the exact labeling of $Q_{2,n-1}$).

By definition, it is easy to see that $f(0, \ldots, 0, 0, 1) = 1$, $f(0, \ldots, 0, 0, m) = m$, and $f(0, \ldots, 0, 1, 1) = m + 1$. We also have the following conditions:
$\max f(R_{m,n}(k)) = f(1, \ldots, 1, 0, \ldots, 0, m)$,
$\min f(R_{m,n}(k)) = f(0, \ldots, 0, 1, \ldots, 1, 1)$,
$\max f(R_{m,n}(k) \cap T_{m,n}(a)) = f(1, \ldots, 1, 0, \ldots, 0, a)$,
$\min f(R_{m,n}(k) \cap T_{m,n}(a)) = f(0, \ldots, 0, 1, \ldots, 1, a)$.

Lemma 7. *Let f, g be the exact labeling of $Q_{m,n}$ and $Q_{m-1,n}$, respectively. Then $g(u_1, \ldots, u_n, i) - g(v_1, \ldots, v_n, i) = f(u_1, \ldots, u_n, j) - f(v_1, \ldots, v_n, j)$ for all $1 \le i \le m-1$ and $1 \le j \le m$ if $u_1 + u_2 + \cdots + u_n = v_1 + v_2 + \cdots + v_n$.*

Proof. It is not hard to check this by the definition of the exact labeling. $\square$

Lemma 8. *Let f be the exact labeling of $Q_{m,n}$. Then*

$$P_f(Q_{m,n}) = m \sum_{i=1}^{n} 2^{n-i} \binom{2i-1}{i} + \binom{m}{2} \binom{2n+1}{n}.$$

Proof. It is trivial that the formula holds for $Q_{1,1}$. We may assume $m \ge 2$. Let g and h_n be the exact labeling of $Q_{m-1,n}$ and $Q_{1,n}$, respectively. Suppose $v = (v_1, \ldots, v_n, a)$. By counting the number of vertices in between those of $N[v]$, we have

$$w_f(v) = \begin{cases} w_g(v) + \binom{n}{\operatorname{rank}(v) - 1}, & \text{if } \operatorname{rank}(v) > 0 \text{ and } a \le m-1, \\ w_g(v), & \text{if } \operatorname{rank}(v) = 0 \text{ and } a \le m-1, \\ w_h(v_1, \ldots, v_n, 1) + (m-1)\binom{n}{\operatorname{rank}(v)}, & \text{if } a = m. \end{cases}$$

Thus,

$$\begin{aligned} P_f(Q_{m,n}) &= \sum_{v=(v_1,\ldots,v_n,a) \in V(Q_{m,n})} w_f(v) \\ &= \sum_{\operatorname{rank}(v)>0, a \le m-1} \left(w_g(v) + \binom{n}{\operatorname{rank}(v)-1} \right) \\ &\quad + \sum_{\operatorname{rank}(v)=0, a \le m-1} w_g(v) + \sum_{a=m} \left(w_{h_n}(v_1, \ldots, v_n, 1) + (m-1) \binom{n}{\operatorname{rank}\,(v)} \right) \\ &= P_g(Q_{m-1.n}) + \sum_{\operatorname{rank}(v)>0, a \le m-1} \binom{n}{\operatorname{rank}(v)-1} + P_{h_n}(Q_{1,n}) + (m-1) \sum_{a=m} \binom{n}{\operatorname{rank}(v)} \\ &= P_g(Q_{m-1,n}) + P_{h_n}(Q_{1,n}) + (m-1) \left[\sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} + \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} \right] \\ &= P_g(Q_{m-1,n}) + P_{h_n}(Q_{1,n}) + (m-1) \binom{2n+1}{n}. \end{aligned}$$

Solving the recurrence relation, we have

$$P_f(Q_{m,n}) = m P_{h_n}(Q_{1,n}) + \binom{m}{2} \binom{2n+1}{n}.$$

Since $Q_{1,n} \cong Q_{2,n-1}$, we have

$$P_{h_n}(Q_{1,n}) = 2P_{h_{n-1}}(Q_{1,n-1}) + \binom{2n-1}{n}.$$

Thus,

$$P_{h_n}(Q_{1,n}) = \sum_{i=1}^{n} 2^{n-i} \binom{2i-1}{i}$$

and

$$P_f(Q_{m,n}) = m \sum_{i=1}^{n} 2^{n-i} \binom{2i-1}{i} + \binom{m}{2}\binom{2n+1}{n}. \quad \square$$

A labeling of $Q_{m,n}$ is *ranking* if it possesses the following properties:

(1) $f(0, 0, \ldots, 0, 1) = 1$.
(2) Suppose $v = (x_1, x_2, \ldots, x_n, a)$, $x_1 = \cdots = x_t = 1$, and $x_{t+1} = \cdots = x_n = 0$ for some $t \geq 0$. Then $f(x_{t+1}, \ldots, x_{n-1}, 1, x_1, \ldots, x_t, 1) = f(v) + 1$ if $a = m$ and $f(x_{t+1}, \ldots, x_n, x_1, \ldots, x_t, a+1) = f(v) + 1$ if $a < m$.
(3) If $v = (x_1, x_2, \ldots, x_n, a)$, $x_1 = \cdots = x_t = 0$, and $x_{t+1} = 1$ for some $t \geq 1$, then $f(x_1, \ldots, x_{t-1}, 1, 0, x_{t+2}, \ldots, x_n, a) = f(v) + 1$.
(4) If $v = (x_1, x_2, \ldots, x_n, a)$, $x_1 = \cdots = x_t = 1$, $x_{t+1} = \cdots = x_{t+s} = 0$, and $x_{t+s+1} = 1$ for some $t, s \geq 1$, then

$$f(x_{t+1}, \ldots, x_{t+s-1}, x_1, \ldots, x_t, 1, 0, x_{t+s+2}, \ldots, x_n, a) = f(v) + 1.$$

Lemma 9. *The exact labeling is ranking.*

Proof. Let $f_{i,j}$ be the exact labeling of $Q_{i,j}$ and let $v = (x_1, x_2, \ldots, x_n, a)$. It is trivial that $f_{m,n}(0, 0, \ldots, 0, 1) = 1$.

Suppose $x_1 = \cdots = x_t = 1$, and $x_{t+1} = \cdots = x_n = 0$ for some $t \geq 0$. Assume $a = m$. Since $\max f_{m,n}(R_{m,n}(t)) = f_{m,n}(x_1, x_2, \ldots, x_n, a)$ and $\min f_{m,n}(R_{m,n}(t+1)) = f_{m,n}(x_{t+1}, \ldots, x_{n-1}, 1, x_1, \ldots, x_t, 1)$, we have $f_{m,n}(x_{t+1}, \ldots, x_{n-1}, 1, x_1, \ldots, x_t, 1) = f_{m,n}(v)+1$. Assume $a < m$. Since $\max f_{m,n}(R_{m,n}(t) \cap T_{m,n}(a)) = f_{m,n}(x_1, x_2, \ldots, x_n, a)$ and $\min f_{m,n}(R_{m,n}(t) \cap T_{m,n}(a+1)) = f_{m,n}(x_{t+1}, \ldots, x_n, x_1, \ldots, x_t, a+1)$, we have

$$f_{m,n}(x_{t+1}, \ldots, x_n, x_1, \ldots, x_t, a+1) = f_{m,n}(v) + 1.$$

If $x_1 = \cdots = x_t = 0$ and $x_{t+1} = 1$ for some $t \geq 1$. Since

$$f_{1,t+1}(0, \ldots, 0, 1, 0, 1) = 3 = f_{1,t+1}(0, \ldots, 0, 0, 1, 1) + 1,$$

by Lemma 7, we have

$$f_{m,n}(x_1, \ldots, x_{t-1}, 1, 0, x_{t+2}, \ldots, x_n, a) = f_{m,n}(v) + 1.$$

Assume $x_1 = \cdots = x_t = 1$, $x_{t+1} = \cdots = x_{t+s} = 0$, and $x_{t+s+1} = 1$ for some $t, s \geq 1$. Then

$$\mathrm{rank}(x_1, \ldots, x_{t+s}, a) + 1 = \mathrm{rank}(x_{t+1}, \ldots, x_{t+s-1}, x_1, \ldots, x_t, 1, a),$$
$$\max f_{1,t+s}(R_{1,t+s}(t)) = f_{1,t+s}(x_1, x_2, \ldots, x_{t+s}, a),$$

and

$$\min f_{1,t+s}(R_{1,t+s}(t+1)) = f_{1,t+s}(x_{t+1}, \ldots, x_{t+s-1}, x_1, \ldots, x_t, 1, 1).$$

Thus,

$$f_{1,t+s+1}(x_1, \ldots, x_{t+s}, 1, 1) + 1 = f_{1,t+s+1}(x_{t+1}, \ldots, x_{t+s-1}, x_1, \ldots, x_t, 1, 0, 1).$$

By Lemma 7, we have

$$f_{m,n}(x_{t+1}, \ldots, x_{t+s-1}, x_1, \ldots, x_t, 1, 0, x_{t+s+2}, \ldots, x_n, 1) = f_n(v) + 1. \quad \square$$

Lemma 10. *Let f be the exact labeling of $Q_{m,n}$. If $u = (x_1, x_2, \ldots, x_n, a)$, $x_1 = \cdots = x_{t-1} = 0$, and $x_t = 1$ for some $t \geq 1$, then $f^{-1}(\min f(N[u])) = (x_1, \ldots, x_{t-1}, 0, x_{t+1}, \ldots, x_n, a)$.*

Proof. Let $w = f^{-1}(\min f(N[u]))$. By the definitions of $Q_{m,n}$ and exact labeling, we have $\mathrm{rank}(w) = \mathrm{rank}(u) - 1$. Assume $\mathrm{rank}(z) = \mathrm{rank}(u) - 1$ and $z \in N[u]$. Then $z = (x_1, \ldots, x_{s-1}, 0, x_{s+1}, \ldots, x_n, a)$ for some $s \geq t$ and $x_s = 1$. Let g be the exact labeling of $Q_{1,s}$. Since $g(x_1, \ldots, x_{s-1}, 0, 1) \geq g(x_1, \ldots, x_{t-1}, 0, x_{t+1}, \ldots x_s, 1)$, by (2)(c) of the definition of exact labeling, it is easy to see that $f(x_1, \ldots, x_{t-1}, 0, x_{t+1}, \ldots, x_n, a) \leq f(z)$. Thus, $w = (x_1, \ldots, x_{t-1}, 0, x_{t+1}, \ldots, x_n, a)$. $\square$

Theorem 11. *The exact labeling of $Q_{m,n}$ is a perfect minimum-neighborhood labeling. Thus,*

$$P(Q_{m,n}) = m \sum_{i=1}^{n} 2^{n-i} \binom{2i-1}{i} + \binom{m}{2} \binom{2n+1}{n}.$$

Proof. By Theorem 5, we need only show that the exact labeling of $Q_{m,n}$ is a direct labeling. It is trivial that the exact labeling of $Q_{m,1}$ is a direct labeling. Let f be the exact labeling of $Q_{m,n-1}$ as well as a perfect minimum-neighborhood labeling of $Q_{m,n-1}$. In procedure DL, we let $(v_i, 1) = 0v_i$ and $(v_i, 2) = 1v_i$. Suppose g is the direct labeling of $Q_{m,n}$ obtained from procedure DL with respect to f. We prove that g is the ranking labeling of $Q_{m,n}$. Since f is the exact labeling, we have $f(0, \ldots, 0, a) = a$. Thus, $g(0, \ldots, 0, a) = a$. Let $v = (x_1, x_2, \ldots, x_n, a)$. Assume $x_1 = \cdots = x_t = 1$ and $x_{t+1} = \cdots = x_n = 0$ for some $t \geq 1$. Let $y = z = 1$ when $a = m$ and $y = a + 1$, $z = x_n$ when $a < m$. Since f is also the ranking labeling of $Q_{m,n-1}$ by Lemma 9, we have

$$f(x_{t+1}, \ldots, x_{n-1}, z, x_2, \ldots, x_t, y) = f(x_2, \ldots, x_n, a) + 1.$$

Since v is of the form $(v_i, 2)$, the next vertex being labeled is in the neighborhood of $(0, x_{t+1}, \ldots, x_{n-1}, z, x_2, \ldots, x_t, y)$. So

$$\begin{aligned} g(x_{t+1}, \ldots, x_{n-1}, z, x_1, \ldots, x_t, y) &= g(0, x_{t+1}, \ldots, x_{n-2}, z, 1, x_2, \ldots, x_t, y) \\ &= g(v) + 1. \end{aligned}$$

Suppose $x_1 = \cdots = x_t = 0$ and $x_{t+1} = 1$ for some $t \geq 1$. If $t \geq 2$, then $f(x_2, \ldots, x_{t-1}, 1, 0, x_{t+2}, \ldots, x_n, a) = f(x_2, \ldots, x_n, a) + 1$. By Lemma 10, we have $\min f(N[(x_2, \ldots, x_n, a)]) = f(x_2, \ldots, x_{t-1}, 0, 0, x_{t+2}, \ldots, x_n, a) = \min f(N[(x_2, \ldots, x_{t-1}, 1, 0, x_{t+2}, \ldots, x_n, a)])$. Thus,

$$g(x_1, \ldots, x_{t-1}, 1, 0, x_{t+2}, \ldots, x_n, a) = g(v) + 1.$$

If $t = 1$, then $f(1, x_3, \ldots, x_n, a) = \max f(N[(0, x_3, \ldots, x_n, a)])$. So,

$$g(1, 0, x_{t+2}, \ldots, x_n, a) = g(v) + 1.$$

If $x_1 = \cdots = x_t = 1$, $x_{t+1} = \cdots = x_{t+s} = 0$, and $x_{t+s+1} = 1$ for some $t, s \geq 1$, then

$$f(x_{t+1}, \ldots, x_{t+s-1}, x_2, \ldots, x_t, 1, 0, x_{t+s+2}, \ldots, x_n, a) = f(x_2, \ldots, x_n, a) + 1.$$

Since v is of the form $(v_i, 2)$, the next vertex being labeled is in the neighborhood of $(0, x_{t+1}, \ldots, x_{t+s-1}, x_2, \ldots, x_t, 1, 0, x_{t+s+2}, \ldots, x_n, a)$. So

$$g(x_{t+1}, \ldots, x_{t+s-1}, x_1, \ldots, x_t, 1, 0, x_{t+s+2}, \ldots, x_n, a) = g(v) + 1. \quad \square$$

Acknowledgements

The first author was supported in part by the National Science Council under grants NSC89-2115-M-259-001. The second author was supported in part by the National Science Council under grants NSC89-2115-M-156-004. The authors would like to thank the referees for many constructive suggestions.

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